Partial Fractions and Frequency Response

APPENDIX H

Dynamic models involve differential equations that are best analyzed using Laplace transform methods. In Chapter 4, the partial fraction method was introduced as a way to invert Laplace transforms, and, more importantly, to establish a basis for determining key system properties like stability and frequency response directly from the transfer function. The methods were explained in Chapters 4 and 10 and applied throughout subsequent chapters. The *proofs* of the methods are provided in this appendix.

H.1 PARTIAL FRACTIONS

The Laplace transform method for solving differential equations could be limited by the availability of entries in Table 4.1, and with so few entries, it would seem that most models could not be solved. However, many complex Laplace transforms can be expressed as a linear combination of a few simple transforms through the use of partial fraction expansion. Once the Laplace transform can be expressed as a sum of simpler elements, each can be inverted individually using the entries in Table 4.1, thus greatly increasing the number of differential equations, that can be solved. More importantly, the application of partial fractions provides generalizations about the forms of solutions to a wide range of differential equation models, and these generalizations enable us to establish important characteristics about a system's time-domain behavior *without* determining the complete transient solution.

The partial fraction expansion can be applied to a Laplace transform that can be expressed as a ratio of polynomials in s. This does not pose a severe limitation, since many models have the form given below for a specified input, $X(s)$.

$$D(s)Y(s) = F(s)X(s) = N(s)$$
$$Y(s) = \frac{N(s)}{D(s)} \tag{H.1}$$

where $Y(s)$ = Laplace transform of the output variable
$X(s)$ = Laplace transform of the input variable
$F(s)$ = Laplace transform of the function $F(t)$; $F(s)X(s)$ is the forcing function
$N(s)$ = numerator polynomial in s of order m
$D(s)$ = denominator polynomial in s of order n, termed the characteristic polynomial

The partial fractions method requires that the order of the denominator be greater than the order of the numerator, i.e., $n > m$; models encountered in process control will satisfy this requirement.

The Laplace transform in equation (H.1) can be expanded into an equivalent expression with simpler individual terms by the application of partial fractions.

Partial Fractions

$$Y(s) = \frac{N(s)}{D(s)} = \frac{C_1}{H_1(s)} + \frac{C_2}{H_2(s)} + \cdots \tag{H.2}$$

$$Y(t) = C_1\mathcal{L}^{-1}\left[\frac{1}{H_1(s)}\right] + C_2\mathcal{L}^{-1}\left[\frac{1}{H_2(s)}\right] + \cdots \tag{H.3}$$

The C_i are constants and the $H_i(s)$ are low-order terms in s which represent the factors of the characteristic polynomial, $D(s) = 0$.

Initially, the C_i's are unknowns in equation (H.2) and must be determined so that the equation is satisfied. There are several ways to determine the constants, and the partial fraction expansions and the resulting Heaviside expansion formula are presented here for three types of factors of the characteristic polynomial; distinct, repeated, and complex.

DISTINCT FACTORS. If the characteristic polynomial has a distinct root at α, the ratio of polynomials can be factored into

$$Y(s) = \frac{N(s)}{D(s)} = \frac{M(s)}{s-\alpha} = \frac{C}{s-\alpha} + R(s) \tag{H.4}$$

with $R(s)$ being the remainder. After multiplying equation (H.4) by $(s-\alpha)$ and setting $s = \alpha$ [resulting in the term $(s-\alpha)H(s)$ being zero], the constant can be determined to be $M(\alpha) = C$. This approach is performed individually for each distinct root, and the function of time, $Y(t)$, is the sum of the inverse Laplace transforms of all individual factors. The expression for distinct factors can be summarized in the following Heaviside expansion which is a generalization of the technique just explained (Churchill, 1972).

$$\mathcal{L}^{-1}\left[\frac{N(s)}{D(s)}\right]_{\left(\substack{n \text{ distinct} \\ \text{factors}}\right)} = \sum_{i=1}^{n} \frac{N(s)|_{s=\alpha_i}}{\left.\frac{d\,D(s)}{ds}\right|_{s=\alpha_i}} e^{\alpha_i t} \tag{H.5}$$

REPEATED FACTORS. A similar partial factor expansion can be applied for "$n+1$" repeated factors, i.e., identical, real roots of the characteristic polynomial $[D(s)]$, as shown below.

$$Y(s) = \frac{N(s)}{D(s)} = \frac{M(s)}{(s-\alpha)^{n+1}} = \frac{C_1}{s-\alpha} + \frac{C_2}{(s-\alpha)^2} + \cdots + \frac{C_{n+1}}{(s-\alpha)^{n+1}} + R(s) \tag{H.6}$$

The coefficients can be determined sequentially by

1. Multiplying equation (H.6) by $(s-\alpha)^{n+1}$ and setting $s = \alpha$ (determining C_{n+1})
2. Multiplying equation (H.6) by $(s-\alpha)^{n+1}$, taking the first derivative with respect to s, and setting $s = \alpha$ (determining C_n)
3. Continuing this procedure (with higher derivatives) until all coefficients have been evaluated

The time-domain function for a repeated factor can be expressed as (Churchill, 1972)

$$\mathcal{L}^{-1}\left[\frac{N(s)}{D(s)}\right]_{\substack{\text{repeated} \\ \text{factor}}} = \frac{1}{n!}\left\{\frac{\partial^n}{\partial s^n}[M(s)e^{st}]\right\}_{s=\alpha} \tag{H.7}$$

where $M(s)$ is defined in equation (H.6).

COMPLEX FACTORS. The final possibility for the factor involves complex factors, and the analysis for a distinct, complex factor is given for the system shown below.

$$Y(s) = \frac{N(s)}{D(s)} = \frac{M(s)}{(s-\alpha)^2 + \omega^2} + R(s) \quad \text{with } \alpha \text{ and } \omega \text{ real} \tag{H.8}$$

The complex roots can be expressed as two distinct roots a $\alpha \pm \omega j$, so that by applying equation (H.2) the Laplace transform and its inverse can be expressed as

$$\frac{N(s)}{D(s)} = \frac{M_1(s)}{s-\alpha+\omega j} + \frac{M_2(s)}{s-\alpha-\omega j} + R(s) \tag{H.9}$$

$$Y(t)_{\substack{\text{complex} \\ \text{factor}}} = [M_1(s)]_{s=\alpha-\omega j} e^{(\alpha-\omega j)t} + [M_2(s)]_{s=\alpha+\omega j} e^{(\alpha+\omega j)t} \tag{H.10}$$

The coefficients in equation (H.10) are complex conjugates and can be expressed as $M_1(+\alpha-\omega j) = (A+Bj)$ and $M_2(+\alpha+\omega j) = (A-Bj)$, respectively. These expressions can be substituted into equation (H.10) to give

$$\begin{aligned} Y(t)_{\substack{\text{complex} \\ \text{factor}}} &= (A+Bj)e^{(\alpha-\omega j)t} + (A-Bj)e^{(\alpha+\omega j)t} \\ &= e^{\alpha t}[A(e^{j\omega t} + e^{-j\omega t}) + jB(e^{-j\omega t} - e^{j\omega t})] \end{aligned} \tag{H.11}$$

Equation (H.11) can be modified to eliminate the complex terms by using the Euler relationships.

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \rightarrow 2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t} \tag{H.12}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \rightarrow 2\sin(\omega t) = j(e^{-j\omega t} - e^{j\omega t}) \tag{H.13}$$

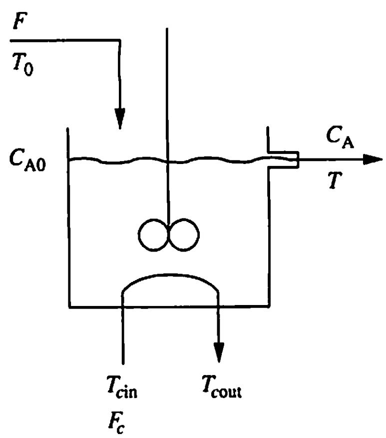
The resulting expression can be used to evaluate the inverse term for a complex conjugate pair of roots of the characteristic polynomial

$$Y(t)_{\substack{\text{complex}\\ \text{factor}}} = 2e^{\alpha t}[A\cos(\omega t) + B\sin(\omega t)] \tag{H.14}$$

The proof of an alternative formulation, along with expressions for repeated complex factors, is available in Churchill (1972).

The application of partial fractions is demonstrated in the following example that includes real and complex roots of the denominator.

EXAMPLE H.1.



For the CSTR modelled in Appendix C, Section C.2, evaluate the inverse Laplace transform of the reactor temperature for a step change in the coolant flow rate.

The original model involved two nonlinear differential equations for the component material and energy balances which were linearized and expressed in deviation variables; these equations are repeated below.

$$\frac{dC'_A}{dt} = a_{11}C'_A + a_{12}T' + a_{13}C'_{A0} + a_{14}F'_c + a_{15}T'_0 + a_{16}F' \tag{C.11}$$

$$\frac{dT'}{dt} = a_{21}C'_A + a_{22}T' + a_{23}C'_{A0} + a_{24}F'_c + a_{25}T'_0 + a_{26}F' \tag{C.12}$$

In this example, the only input variable which changes is the coolant flow which experiences a step, so that $C'_{A0}(s) = T'_0(s) = F'(s) = 0$. The Laplace transforms of equations (C.11) and (C.12) can be taken to give

$$sC'_A(s) = a_{11}C'_A(s) + a_{12}T'(s) + a_{14}F'_c(s) \tag{H.15}$$

$$sT'(s) = a_{21}C'_A(s) + a_{22}T'(s) + a_{24}F'_c(s) \tag{H.16}$$

Equations (H.15) and (H.16) can be combined algebraically. First, equation (H.15) is rearranged to solve for $C'_A(s) = a_{12}T'(s)/(s - a_{11})$, since $a_{14} = 0$; this term is then substituted into equation (H.16) to give

$$T'(s) = \frac{a_{24}s + (a_{21}a_{14} - a_{24}a_{11})}{s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21})} F'_c(s) \tag{H.17}$$

When the numerical values are substituted into equation (H.17) using the CSTR data in Section C.2, the result is

$F'_c(s) = -1/s$	$a_{11} = -7.55$	$a_{12} = -0.0931$	$a_{14} = 0.0$
	$a_{21} = 852.02$	$a_{22} = 5.77$	$a_{24} = -6.07$

$$T'(s) = \frac{(-1)[-6.07(s) - 45.83]}{s(s^2 + 1.79s + 35.80)} \tag{H.18}$$

Partial fraction expansion requires the roots of the characteristic polynomial, which are $-0.894 \pm 5.92j$ and 0.0; thus, two factors are complex. The inverse transform for the complex factors can be determined by using equations (H.11) and (H.14) with $A = -0.64$, $B = 0.42$, $\alpha = -0.894$, and $\omega = 5.94$.

$$M_1(s)|_{s=\alpha-\omega j} = \left[\frac{(-1)(-6.07s - 45.83)}{(s + 0.894 - 5.92j)(s)}\right]_{s=-0.894-5.92j} = -0.64 + 0.42j \quad \text{(H.19)}$$

$$T'(t)_{\substack{\text{complex}\\\text{factor}}} = 2e^{-0.894t}[-0.64\cos(5.92t) + 0.42\sin(5.92t)] \quad \text{(H.20)}$$

The single distinct factor can be inverted using equation (H.5).

$$M(s)|_{s=0} = \left[\frac{(-1)(-6.07s - 45.83)}{s^2 + 1.789s + 35.80}\right]_{s=0} = 1.28 \quad \text{(H.21)}$$

$$T'(t)_{\text{distinct}} = 1.28e^{0t} = 1.28 \quad \text{(H.22)}$$

The complete inverse transform is the sum of the two functions.

$$T'(t) = 1.28 + 2e^{-0.894t}[-0.64\cos(5.92t) + 0.42\sin(5.92t)] \quad \text{(H.23)}$$

The solution to the linear approximation in equation (H.23) is a damped oscillation. This underdamped behavior did not occur in the simple processes modelled in Chapters 3 and 4. A comparison of the solutions to the linearized and nonlinear equations in Figure H.1 shows how the linearized model represents the essential characteristics of the true process response. Naturally, the accuracy of the linear approximation depends on the size of the input change, with the accuracy improving as the input magnitude decreases.

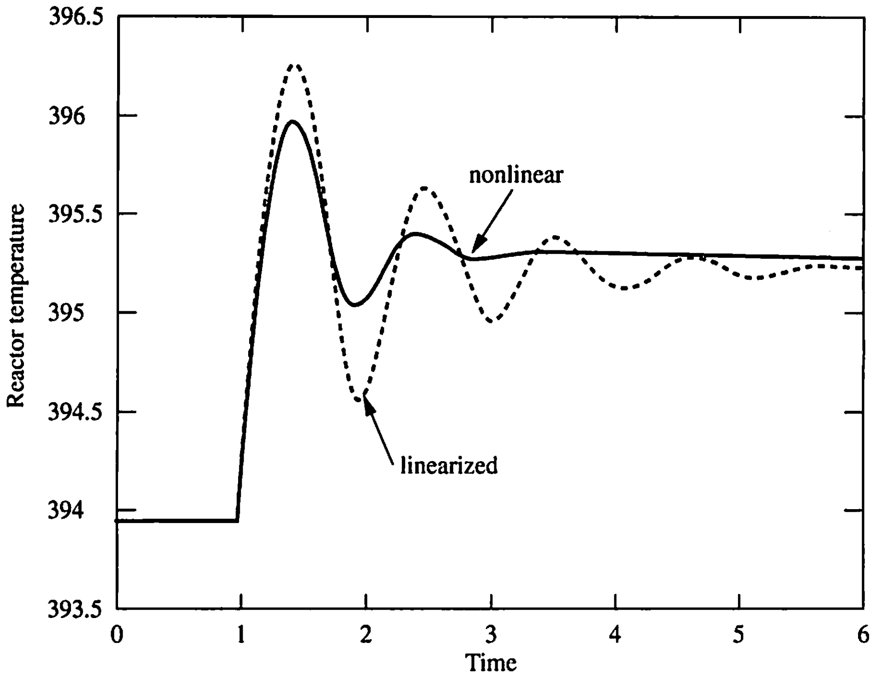


FIGURE H.1

Linearized and nonlinear responses for a step of −1 m³/min in coolant flow at t = 1 min for the CSTR in Example H.1.

H.2 FREQUENCY RESPONSE

Frequency response is defined as the output variable behavior resulting from a sine input variation after short-term transients become negligible. Frequency response is important in determining the stability and control performance of linear dynamic systems, and it is used extensively in process control. The simplified method for evaluating the frequency response used in process control involves determining the amplitude ratio and phase angle from the transfer function with $s = \omega j$; the *proof* for this method is presented in this section. We begin with a general expression for the frequency response; the following equation gives output $Y(s)$ of a linear system with a transfer function $G(s)$ and a sine input forcing $X(s)$ with magnitude A and frequency ω.

$$Y(s) = G(s)X(s) = G(s)A\frac{\omega}{s^2+\omega^2} \tag{H.24}$$

The transfer function is assumed to be a ratio of polynomials, so that the solution can be analyzed using a partial fractions expansion of the right-hand side of equation (H.24), as explained in the previous section. The general form of the solution to equation (H.24) can be determined by accounting for all poles (roots of the denominator) whether real distinct, real repeated, or complex.

$$\begin{aligned} Y(t) &= A_1 e^{\alpha_1 t} + \cdots + (B_1 + B_2 t + B_3 t^2 + \cdots)e^{\alpha_p t} \\ &\quad + [C_1 \cos(\omega t) + C_2 \sin(\omega t)]e^{\alpha_q t} + \cdots + D_1 e^{-j\omega t} + D_2 e^{j\omega t} \end{aligned} \tag{H.25}$$

The final two terms in equation (H.25) account the additional poles from the sine input. All but the last two terms tend toward zero as time increases, as long as the system is stable, i.e., $\text{Re}(\alpha_i) < 0$ for all i. Thus, only the last two terms in equation (H.25) affect the output behavior after a long time, i.e., which is the definition of the frequency response. The constants for the last two terms can be evaluated using the partial fractions method for distinct roots, $\alpha_1 = -j\omega$ and $\alpha_2 = +j\omega$.

$$D_1 = \left[G(s)\frac{A\omega}{(s-j\omega)}\right]_{s=-j\omega} = G(s)|_{s=-j\omega}\frac{A}{-2j} = -A\frac{G(-j\omega)}{2j} \tag{H.26}$$

$$D_2 = \left[G(s)\frac{A\omega}{(s+j\omega)}\right]_{s=+j\omega} = G(s)|_{s=+j\omega}\frac{A}{+2j} = A\frac{G(j\omega)}{+2j} \tag{H.27}$$

Since only these terms affect the long-time behavior, the output can be expressed as (with the subscript FR for the frequency response)

$$Y_{\text{FR}}(t) = -\frac{A}{2j}G(-j\omega)e^{-j\omega t} + \frac{A}{2j}G(j\omega)e^{j\omega t} \tag{H.28}$$

Any transfer function, which involves complex numbers, can be expressed in polar form using

$$G(j\omega) = |G(j\omega)|e^{j\phi} \quad \text{with} \quad \phi = \angle G(j\omega) = \tan^{-1}\left\{\frac{\text{Im}[G(j\omega)]}{\text{Re}[G(j\omega)]}\right\} \tag{H.29}$$

Equation (H.28) can be expressed in polar form using equation (H.29) to give

$$Y_{\text{FR}}(t) = -\frac{A}{2j}|G(\omega j)|e^{-(\omega t+\phi)j} + \frac{A}{2j}|G(\omega j)|e^{(\omega t+\phi)j} \tag{H.30}$$

This result, along with Euler's identity to convert the exponential expressions to a sine, gives the final expression for the frequency response of a general linear system.

$$Y_{FR}(t) = A \mid G(j\omega) \mid \sin(\omega t + \phi) = B \sin(\omega t + \phi) \qquad \text{(H.31)}$$

Thus, the output variable $Y(t)$ is also a sine with (1) the same frequency ω as the input, (2) an amplitude B, and (3) a phase shift of ϕ from the input. The simplified method for evaluating the output variable frequency response of a linear dynamic system proved in this section is to set the Laplace variable $s = \omega j$ in the transfer function and evaluate the magnitude and phase, which provide the amplitude ratio and phase angle, as summarized below.

$$\text{Amplitude ratio} = B/A = |G(\omega j)| \qquad \text{(H.32)}$$

$$\text{Phase angle} = \phi = \angle G(\omega j) \qquad \text{(H.33)}$$

This result proves that an amazing amount of information about the dynamic behavior of a linear system can be determined without the effort of evaluating its inverse Laplace transform.

REFERENCE

Churchill, R., *Operational Mathematics*, McGraw-Hill, New York, 1972.