Discrete Models for Digital Control

APPENDIX F

The chemical processes considered in this book involve continuous variables and can be modelled using algebraic and differential equations. Also, the control calculations have been introduced as equations involving continuous variables, which can be implemented using electronic or pneumatic analog calculating equipment. When all elements in the feedback loop are continuous, the system can be described using transfer functions involving Laplace transforms; this allows powerful analysis tools to be applied in determining the stability and performance of control systems. However, most control calculations are now implemented using digital computers, which introduce discrete equations in the control system. If the digital calculations are executed rapidly compared with the process dynamics, the analysis of continuous systems provides an accurate approximation of the dynamic behavior.

Because the controller is implemented in digital form, it is important that the engineer understand the digital forms of the models and control calculations used in this book. The major applications of digital calculations are summarized below.

Chapter 3: numerical solutions of differential equations using Euler or Runge-Kutta methods
Chapter 6: least squares fitting of parameters in dynamic models
Chapter 11: digital formulation of the PID controller
Chapter 12: implementation issues for digital control
Chapter 15: lead/lag elements for feedforward control
Chapter 19: digital formulation of the model predictive controller
Chapter 21: decoupling of multiple PID controllers
Chapter 23: dynamic matrix control

The material in Chapters 11, 12, and 23 is self-contained and will not be repeated here. The topics covered in this appendix involve the discrete forms of simple models used in process control. The process models will be represented using X as the input (cause) and Y as the output (effect). The current sampled value of a variable will be designated by the subscript n, the previous value by $n-1$, and so forth, with the sample period being constant at Δt.

F.1 ■ GAIN

The output of a gain is simply calculated as a proportion K of the input:

$$Y_n = K X_n \tag{F.1}$$

F.2 ■ DEAD TIME

Dead time is simulated as a delay table of length Γ, which is an integer equal to $\theta/\Delta t$. At each time step, the model is executed by moving the past values to the location representing the next oldest value; the oldest value is discarded, and the previous value is placed in the table in the location of the most recent value. This calculation is summarized in Table F.1 with a delay table of length 4 (e.g., a dead time of 2 units of time and a sample period of 0.5), for eight time steps. The input is a pulse with a duration of two time steps.

This approach is simple to program and prevents the need to store all past data, because the table needs to store data for only the length of the dead time. More computationally efficient implementations move only one data point each execution and use an additional variable (pointer) to indicate the position of the oldest data in the table.

F.3 ■ FIRST-ORDER SYSTEM

Material and energy balances yield first-order differential equations, and the most common model is first-order with dead time. Thus, the first-order models are used frequently.

$$\tau \frac{dY}{dt} = K X - Y \tag{F.2}$$

The continuous model can be expressed as a discrete model by assuming that the input is constant at the value of X_{n-1} over the period t_{n-1} to t_n (or 0.0 to

TABLE F.1
Example delay table for simulating dead time

Sample number, n	1	2	3	4	5	6	7	8
Input, X_n	0	1	1	0	0	0	0	0
Table entry 1, X_{n-1} (most recent)	0	0	1	1	0	0	0	0
Table entry 2, X_{n-2}	0	0	0	1	1	0	0	0
Table entry 3, X_{n-3}	0	0	0	0	1	1	0	0
Table entry 4, X_{n-4} (oldest)	0	0	0	0	0	1	1	0
Output, $Y_n = X_{n-4}$	0	0	0	0	0	1	1	0

Δt). Then, integration of the differential equation from the initial condition Y_{n-1} can be performed to determine the value at Y_n. The solution can be determined using the integrating factor or Laplace transform; here the Laplace transform is demonstrated.

$$\tau s Y(s) - \tau Y_{n-1} = K X(s) - Y(s) \tag{F.3}$$

$$Y(s) = \frac{K X(s)}{\tau s + 1} + \frac{\tau Y_{n-1}}{\tau s + 1} \tag{F.4}$$

Note that $Y(t_{n-1}) = Y_{n-1}$. The input is evaluated as X_{n-1}/s, and the inverse transform can be taken to give

$$Y_n = K(1 - e^{-\Delta t/\tau})X_{n-1} + e^{-\Delta t/\tau} Y_{n-1} \tag{F.5}$$

Equation (F.5) gives exact sampled values if the process is truly first-order and the input is constant over the period. If the input changes during the period, then the use of X_{n-1} as a constant results in an approximation. An alternative, approximate model can be derived by approximating the derivative as a difference, $dY/dt \approx (Y_{n+1} - Y_n)/\Delta t$. This results in

$$Y_n = K\left(\frac{\Delta t}{\tau}\right) X_{n-1} + \left(1 - \frac{\Delta t}{\tau}\right) Y_{n-1} \tag{F.6}$$

Equations (F.5) and (F.6) give very similar results when the sample period is small compared with the time constant. For example, when $\Delta t/\tau = 0.05$, $e^{-\Delta t/\tau} = 0.951$ and $(1 - \Delta t/\tau) = 0.95$.

These discrete models can be used to represent a process and to implement a first-order filter, as described in Chapter 12. Also, the gain, dead time, and first-order discrete models can be combined to give for first-order with dead time:

$$Y_n = e^{-\Delta t/\tau} Y_{n-1} + K(1 - e^{-\Delta t/\tau})X_{n-\Gamma-1} \tag{F.7}$$

Equation (F.7) is employed when using least squares to determine the values of the model parameters from discrete (sampled), empirical input-output data; it is also used as the prediction model in the IMC and Smith predictor model predictive control systems.

F.4 ▣ LEAD/LAG

The final discrete control calculation in this appendix is the lead/lag algorithm, which is as follows for a continuous system:

$$Y(s) = \frac{T_{\text{ld}} s + 1}{T_{\text{lg}} s + 1} X(s) \tag{F.8}$$

A straightforward manner for developing an approximate discrete lead/lag is to replace each "derivative," which is the product of the Laplace variable s and a variable, with its finite difference approximation. This gives

$$T_{\text{lg}}\left(\frac{Y_n - Y_{n-1}}{\Delta t}\right) + Y_n = T_{\text{ld}}\left(\frac{X_n - X_{n-1}}{\Delta t}\right) + X_n \tag{F.9}$$

This can be rearranged to give

$$Y_n = \left(\frac{\frac{T_{\text{lg}}}{\Delta t}}{\frac{T_{\text{lg}}}{\Delta t} + 1}\right) Y_{n-1} + \left(\frac{\frac{T_{\text{ld}}}{\Delta t} + 1}{\frac{T_{\text{lg}}}{\Delta t} + 1}\right) X_n - \left(\frac{\frac{T_{\text{ld}}}{\Delta t}}{\frac{T_{\text{lg}}}{\Delta t} + 1}\right) X_{n-1} \tag{F.10}$$

Equation (F.10) is used in feedforward controllers, as described in Chapter 15, and decouplers (another form of feedforward), as described in Chapter 21; it is also used for the combined IMC filter and controller, $G_f(s)G_{cp}(s)$, for a controller whose model has an invertible process factor that is first-order and with a filter that is first-order.

The discrete models of dynamic systems are in the form of difference equations, in which the current values of a variable can be expressed as a function of the last few values of the output and the input(s). In this appendix the difference equations have been formulated to calculate the nth sampled value. Any equation of this form can be modified to calculate, for example, the $(n+1)$th value. This can be done by substituting $n-1=m$ in the expressions; the result for the first-order system is

$$Y_{m+1} = K(1 - e^{-\Delta t/\tau})X_m + e^{-\Delta t/\tau}Y_m \tag{F.11}$$

Equations (F.5) and (F.11) are equivalent, and both formulations are commonly used, so the reader should be acquainted with both.