Approximate Dynamic Models

APPENDIX D

D.1 METHOD OF MOMENTS

Real processes have complex dynamic responses and require models with many parameters to be characterized accurately. However, the engineer often seeks a simple model with few parameters to describe the main aspects of the dynamic behavior. Examples throughout this book demonstrate that the first-order-with-dead-time model is adequate for the process control analysis of many, but not all, processes. In this section a method is developed for determining a few parameters that can be used to fit a model to the expected dynamic behavior; this is the *method of moments*. The application of the method of moments described in this appendix was demonstrated by Paynter and Takahashi (1956) and Gibilaro and Lee (1969).

The basic approach is to evaluate several *moments* of the output behavior and use these to characterize the dynamic behavior. Thus, the first step is to define a moment.

The nth **moment** of a variable $Y(t)$ is

$$M_n = \frac{\int_0^\infty t^n Y(t)\,dt}{\int_0^\infty Y(t)\,dt} \tag{D.1}$$

Further moments are usually defined with respect to the first moment, which is the *mean*; thus, the moments of the variable $Y(t)$ about its mean are

$$T_n = \frac{\int_0^\infty (t - M_1)^n Y(t)\, dt}{\int_0^\infty Y(t)\, dt} \tag{D.2}$$

Given a function $Y(t)$ or a set of data Y, the integrals in equations (D.1) and (D.2) can be evaluated as long as they are bounded.

The moments can also be evaluated from the Laplace transform of a variable in a particularly simple manner, which is the application of moments in this book. The development begins with the input-output model of a single-variable system in transfer function form.

$$Y(s) = G(s)X(s) \tag{D.3}$$

with $X(s)$ being the input, $Y(s)$ the output, and $G(s)$ the transfer function, as defined in equation (4.45). The moment of the output variable will be evaluated for a unit impulse input, for which $X(s) = 1$ and all integrals in the moment equations are bounded. From the definition of the Laplace transform and equation (D.3),

$$\int_0^\infty e^{-st} Y(t)\, dt = Y(s) = G(s)X(s) = G(s) \tag{D.4}$$

Now, it is shown that any moment of an output in response to a unit impulse can be evaluated directly from the transfer function, using the result in equation (D.4) to evaluate the numerator and denominator of equation (D.1).

$$\int_0^\infty Y(t)\, dt = G(s)|_{s=0} \tag{D.5}$$

$$\int_0^\infty t^n Y(t)\, dt = (-1)^n \left(\frac{d^n}{ds^n} G(s) \right)_{s=0} \tag{D.6}$$

Equation (D.6) is verified using the results from equation (D.4).

$$\begin{aligned} (-1)^n \left(\frac{d^n}{ds^n} G(s) \right)_{s=0} &= \left((-1)^n \int_0^\infty (-t)^n e^{-st} Y(t)\, dt \right)_{s=0} \\ &= \int_0^\infty t^n Y(t)\, dt \end{aligned} \tag{D.7}$$

The method of moments is used in this book for one important application: determining the *characteristic time* of a process. The first moment is used as the characteristic time to "time-scale" the dynamic responses in the dimensional analysis presented in the tuning correlations in Chapter 9. For example, the first moment is evaluated for a first-order-with-dead-time process model to be

$$\int_0^\infty Y(t)\, dt = \left(\frac{K_p e^{-\theta s}}{\tau s + 1} \right)_{s=0} = K_p \tag{D.8}$$

$$\begin{aligned} \int_0^\infty t Y(t)\, dt &= (-1) \left(\frac{d}{ds} \frac{K_p e^{-\theta s}}{\tau s + 1} \right)_{s=0} \\ &= (-1) \left(\frac{-\theta K_p e^{-\theta s}}{\tau s + 1} + \frac{-\tau K_p e^{-\theta s}}{(\tau s + 1)^2} \right)_{s=0} = K_p(\theta + \tau) \end{aligned} \tag{D.9}$$

$$M_1 = \frac{K_p(\theta + \tau)}{K_p} = \theta + \tau \tag{D.10}$$

This result was used by Jeffreson (1976) in performance correlations.

The sum of the dead time and time constant is also the time at which the output response for a step in the manipulated variable reaches 63% of its final value ($t_{63\%}$) for the first-order-with-dead-time model. As a rough approximation, the first moment of many common transfer functions in the book can be used as an estimate of $t_{63\%}$. The first moment for a transfer function with dead time, multiple first-order numerator terms, and multiple first-order denominator terms is evaluated as follows:

$$\int_0^\infty tY(t)\,dt = (-1)\left(\frac{d}{ds}\frac{K_p e^{-\theta s}\prod_{j=0}^{m}(\tau_{\text{ld}j}s+1)}{\prod_{k=0}^{q}(\tau_k s+1)}\right)_{s=0}$$

$$= \theta\left(\frac{K_p e^{-\theta s}\prod_{j}(\tau_{\text{ld}j}s+1)}{\prod_{k}(\tau_k s+1)}\right)_{s=0}$$

$$+\sum_{r=1}^{m} -\tau_{\text{ld}r}\left(\frac{K_p e^{-\theta s}\prod_{j}^{j\neq r}(\tau_{\text{ld}j}s+1)}{\prod_{k}(\tau_k s+1)}\right)_{s=0} \tag{D.11}$$

$$+\sum_{\text{kk}=0}^{q}\left[\tau_{\text{kk}}\prod_{k=1}^{k\neq \text{kk}}(\tau_k s+1)\right]_{s=0}\left(\frac{K_p e^{-\theta s}\prod_{j}(\tau_{\text{ld}j}s+1)}{\left(\prod_{k}(\tau_k s+1)\right)^2}\right)_{s=0}$$

$$= K_p\left(\theta + \sum_{k=0}^{q}\tau_k - \sum_{j=0}^{m}\tau_{\text{ld}j}\right)$$

$$M_1 = \frac{K_p}{K_p}\left(\theta + \sum_{k=0}^{q}\tau_k - \sum_{j=0}^{m}\tau_{\text{ld}j}\right) \tag{D.12}$$

This is the basis for the approximation given in Chapter 5 that $t_{63\%}$ is approximately equal to the sum of the dead times and time constants for a series of noninteracting first-order-with-dead-time systems. This approximation is useful for estimating the general time for a complex series system to respond, but it does not give sufficient information in itself to design or tune controllers.

An additional application for the method of moments is in estimating the parameters in a simple model based on the parameters in a more complex model. In this approach, several moments of the simple and more complex models are determined analytically, and the unknown parameters are determined for the simple model. Naturally, one linearly independent moment equation is required for each parameter. This is demonstrated as follows by determining the parameters for a first-order-with-dead-time model based on a *known* second-order-with-dead-time model.

SECOND-ORDER MODEL.

$$G(s) = \frac{K_{p2}e^{-\theta_2 s}}{(\tau_{21}s + 1)(\tau_{22}s + 1)}$$

Unit impulse $\int_0^\infty Y(t)\,dt$: K_{p2}

First moment: $(\theta_2 + \tau_{21} + \tau_{22})$

Second moment: $\theta_2^2 + 2\theta(\tau_{21} + \tau_{22}) - 2\tau_{21}\tau_{22} + 2(\tau_{21} + \tau_{22})^2$

FIRST-ORDER MODEL:

$$\frac{K_{p1}e^{-\theta_1 s}}{\tau_1 s + 1}$$

Unit impulse $\int_0^\infty Y(t)\,dt$: K_{p1}

First moment: $(\theta_1 + \tau_1)$

Second moment: $\theta_1^2 + 2\theta_1\tau_1 + 2\tau_1^2$

These equations can be applied to the second-order-with-dead-time model in question 6.5 to answer part (*b*) of the question: what is an approximate first-order-with-dead-time model? The results are summarized as follows:

Second-order:

$$\frac{T_4(s)}{T_2(s)} = \frac{1.87e^{-2.6s}}{(2s + 1)(2.7s + 1)}$$

Equating the moments gives

$$K_{p1} = 1.87$$
$$M_1 = \theta_1 + \tau_1 = 2.6 + 2.0 + 2.7 = 7.3$$
$$M_2 = \theta_1^2 + 2\theta_1\tau_1 + 2\tau_1^2 = 64.98$$

giving

$$\theta_1 = 3.3 \qquad \tau_1 = 4.0$$

Approximate first-order:

$$\frac{T_4(s)}{T_2(s)} = \frac{1.87e^{-3.3s}}{4s + 1}$$

D.2 ▣ PADÉ DEAD TIME APPROXIMATIONS

Some control analysis methods are designed for process models that do not contain dead time; i.e., the transfer function models must be ratios of polynomials in the numerator and denominator. To meet this requirement, the dead time in a transfer function model ($e^{-\theta s}$) must be replaced by an approximation. One straightforward approach would be to expand the dead time in a Taylor series. However, better approximations are available using the Padé approximations (Truxal, 1955). The first-order Padé approximation is given in the following:

$$\text{Padé approximation:} \quad e^{-\theta s} \cong \frac{1-(\theta/2)s}{1+(\theta/2)s} \tag{D.13}$$

As an example, the Padé approximation is applied to the simple first-order-with-dead-time transfer function model.

$$\text{Exact model:} \quad \frac{Y(s)}{X(s)} = G(s) = \frac{1.0e^{-5s}}{5s+1} \tag{D.14}$$

An approximate model without the exponential term can be determined by substituting the Padé approximation for the dead time to yield the following:

$$\text{Approximate model:} \quad \frac{Y(s)}{X(s)} = G(s) = \frac{1.0}{5s+1}\frac{(1-2.5s)}{(1+2.5s)} \tag{D.15}$$

The dynamic responses for the exact and approximate models are now compared. The time-domain responses of the output, Y, to a step in the input variable, X, are given in Figure D.1 for both the exact and approximate models. The Padé model

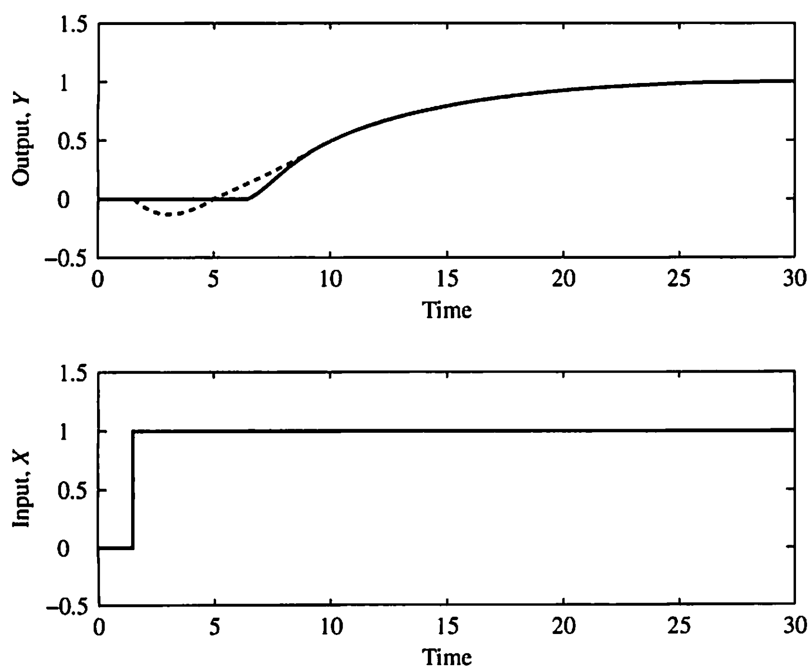

FIGURE D.1

Step responses of exact first-order-with-dead-time model (solid) and first-order-with-Padé-approximation model (dashed).

shows an approximate delay, but it experiences an inverse response not present in the output of the exact model.

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